

CH5 - COMPLEX NUMBERS AND QUADRATIC EQUATIONS

Exercise 5.1 Page No: 103

Express each of the complex numbers given in Exercises 1 to 10 in the form $a + ib$.

1. $(5i)(-3/5i)$

Solution:

$$\begin{aligned}(5i)(-3/5i) &= 5 \times (-3/5) \times i^2 \\ &= -3 \times -1 [i^2 = -1] \\ &= 3\end{aligned}$$

Hence,

$$(5i)(-3/5i) = 3 + i0$$

2. $i^9 + i^{19}$

Solution:

$$\begin{aligned}i^9 + i^{19} &= (i^2)^4 \cdot i + (i^2)^9 \cdot i \\ &= (-1)^4 \cdot i + (-1)^9 \cdot i \\ &= 1 \times i + -1 \times i \\ &= i - i \\ &= 0\end{aligned}$$

Hence,

$$i^9 + i^{19} = 0 + i0$$

3. i^{-39}

Solution:

$$i^{-39} = 1/i^{39} = 1/i^{4 \times 9 + 3} = 1/(i^9 \times i^3) = 1/i^3 = 1/(-i) [i^4 = 1, i^3 = -i \text{ and } i^2 = -1]$$

Now, multiplying the numerator and denominator by i we get

$$\begin{aligned}i^{-39} &= 1 \times i / (-i \times i) \\ &= i / 1 = i\end{aligned}$$

Hence,

$$i^{-39} = 0 + i$$

4. $3(7 + i7) + i(7 + i7)$

Solution:

$$\begin{aligned}3(7 + i7) + i(7 + i7) &= 21 + i21 + i7 + i^2 7 \\ &= 21 + i28 - 7 [i^2 = -1] \\ &= 14 + i28\end{aligned}$$

Hence,

$$3(7 + i7) + i(7 + i7) = 14 + i28$$

$$5. (1 - i) - (-1 + i6)$$

Solution:

$$(1 - i) - (-1 + i6) = 1 - i + 1 - i6 \\ = 2 - i7$$

Hence,

$$(1 - i) - (-1 + i6) = 2 - i7$$

6.

$$\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$$

Solution:

$$\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$$

$$= \frac{1}{5} + \frac{2}{5}i - 4 - \frac{5}{2}i$$

$$= \left(\frac{1}{5} - 4\right) + i\left(\frac{2}{5} - \frac{5}{2}\right)$$

$$= \frac{-19}{5} + i\left(\frac{-21}{10}\right)$$

$$= \frac{-19}{5} - \frac{21}{10}i$$

Hence,

$$\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right) = \frac{-19}{5} - \frac{21}{10}i$$

$$7. \left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right)$$

Solution:

$$\begin{aligned}& \left[\left(\frac{1}{3} + i\frac{7}{3} \right) + \left(4 + i\frac{1}{3} \right) \right] - \left(-\frac{4}{3} + i \right) \\&= \frac{1}{3} + \frac{7}{3}i + 4 + \frac{1}{3}i + \frac{4}{3} - i \\&= \left(\frac{1}{3} + 4 + \frac{4}{3} \right) + i \left(\frac{7}{3} + \frac{1}{3} - 1 \right) \\&= \frac{17}{3} + i\frac{5}{3}\end{aligned}$$

Hence,

$$\left[\left(\frac{1}{3} + i\frac{7}{3} \right) + \left(4 + i\frac{1}{3} \right) \right] - \left(-\frac{4}{3} + i \right) = \frac{17}{3} + i\frac{5}{3}$$

8. $(1 - i)^4$

Solution:

$$\begin{aligned}(1 - i)^4 &= [(1 - i)^2]^2 \\&= [1 + i^2 - 2i]^2 \\&= [1 - 1 - 2i]^2 \quad [i^2 = -1] \\&= (-2i)^2 \\&= 4(-1) \\&= -4\end{aligned}$$

Hence, $(1 - i)^4 = -4 + 0i$

9. $(1/3 + 3i)^3$

Solution:

$$\begin{aligned}\left(\frac{1}{3} + 3i \right)^3 &= \left(\frac{1}{3} \right)^3 + (3i)^3 + 3 \left(\frac{1}{3} \right)(3i) \left(\frac{1}{3} + 3i \right) \\&= \frac{1}{27} + 27i^3 + 3i \left(\frac{1}{3} + 3i \right) \\&= \frac{1}{27} + 27(-i) + i + 9i^2 \quad [i^3 = -i] \\&= \frac{1}{27} - 27i + i - 9 \quad [i^2 = -1] \\&= \left(\frac{1}{27} - 9 \right) + i(-27 + 1) \\&= \frac{-242}{27} - 26i\end{aligned}$$

Hence, $(1/3 + 3i)^3 = -242/27 - 26i$

10. $(-2 - 1/3i)^3$

Solution:

$$\begin{aligned}
 \left(-2 - \frac{1}{3}i\right)^3 &= (-1)^3 \left(2 + \frac{1}{3}i\right)^3 \\
 &= - \left[2^3 + \left(\frac{i}{3}\right)^3 + 3(2)\left(\frac{i}{3}\right)\left(2 + \frac{i}{3}\right) \right] \\
 &= - \left[8 + \frac{i^3}{27} + 2i \left(2 + \frac{i}{3}\right) \right] \\
 &= - \left[8 - \frac{i}{27} + 4i + \frac{2i^2}{3} \right] \quad [i^3 = -i] \\
 &= - \left[8 - \frac{i}{27} + 4i - \frac{2}{3} \right] \quad [i^2 = -1] \\
 &= - \left[\frac{22}{3} + \frac{107i}{27} \right] \\
 &= -\frac{22}{3} - \frac{107}{27}i
 \end{aligned}$$

Hence,

$$(-2 - 1/3i)^3 = -22/3 - 107/27i$$

Find the multiplicative inverse of each of the complex numbers given in Exercises 11 to 13.

11. $4 - 3i$

Solution:

Let's consider $z = 4 - 3i$

Then,

$$= 4 + 3i \text{ and}$$

$$|z|^2 = 4^2 + (-3)^2 = 16 + 9 = 25$$

Thus, the multiplicative inverse of $4 - 3i$ is given by z^{-1}

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3}{25}i$$

12. $\sqrt{5} + 3i$

Solution:

Let's consider $z = \sqrt{5} + 3i$

$$\text{Then, } \bar{z} = \sqrt{5} - 3i \text{ and } |z|^2 = (\sqrt{5})^2 + 3^2 = 5 + 9 = 14$$

$$|z|^2 = (\sqrt{5})^2 + 3^2 = 5 + 9 = 14$$

Thus, the multiplicative inverse of $\sqrt{5} + 3i$ is given by z^{-1}

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5}-3i}{14} = \frac{\sqrt{5}}{14} - \frac{3i}{14}$$

13. $-i$

Solution:

Let's consider $z = -i$

Then, $\bar{z} = i$ and $|z|^2 = 1^2 = 1$

Thus, the multiplicative inverse of $-i$ is given by z^{-1}

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{i}{1} = i$$

14. Express the following expression in the form of $a + ib$:

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$$

Solution:

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$$

$$= \frac{(3)^2 - (i\sqrt{5})^2}{\sqrt{3} + \sqrt{2}i - \sqrt{3} + \sqrt{2}i}$$

$$[(a+b)(a-b) = a^2 - b^2]$$

$$= \frac{9 - 5i^2}{2\sqrt{2}i}$$

$$= \frac{9 - 5(-1)}{2\sqrt{2}i}$$

$$[i^2 = -1]$$

$$= \frac{9+5}{2\sqrt{2}i} \times \frac{i}{i}$$

$$= \frac{14i}{2\sqrt{2}i^2}$$

$$= \frac{14i}{2\sqrt{2}(-1)}$$

$$= \frac{-7i}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{-7\sqrt{2}i}{2}$$

Hence,

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})} = 0 + \frac{-7\sqrt{2}i}{2}$$

Exercise 5.2 Page No: 108

Find the modulus and the arguments of each of the complex numbers in Exercises 1 to 2.

1. $z = -1 - i\sqrt{3}$

Solution:

Given,

$$z = -1 - i\sqrt{3}$$

$$\text{Let } r \cos \theta = -1 \text{ and } r \sin \theta = -\sqrt{3}$$

On squaring and adding, we get

$$(r \cos \theta)^2 + (r \sin \theta)^2 = (-1)^2 + (-\sqrt{3})^2$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 3$$

$$r^2 = 4 \quad [\cos^2 \theta + \sin^2 \theta = 1]$$

$$r = \sqrt{4} = 2 \quad [\text{Conventionally, } r > 0]$$

Thus, modulus = 2

So, we have

$$2 \cos \theta = -1 \text{ and } 2 \sin \theta = -\sqrt{3}$$

$$\cos \theta = \frac{-1}{2} \text{ and } \sin \theta = \frac{-\sqrt{3}}{2}$$

As the values of both $\sin \theta$ and $\cos \theta$ are negative, θ lies in III Quadrant,

$$\text{Argument} = -\left(\pi - \frac{\pi}{3}\right) = \frac{-2\pi}{3}$$

Therefore, the modulus and argument of the complex number $-1 - \sqrt{3}i$ are 2 and $\frac{-2\pi}{3}$ respectively.**2. $z = -\sqrt{3} + i$** **Solution:**

Given,

$$z = -\sqrt{3} + i$$

$$\text{Let } r \cos \theta = -\sqrt{3} \text{ and } r \sin \theta = 1$$

On squaring and adding, we get

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = (-\sqrt{3})^2 + 1^2$$

$$r^2 = 3 + 1 = 4 \quad [\cos^2 \theta + \sin^2 \theta = 1]$$

$$r = \sqrt{4} = 2 \quad [\text{Conventionally, } r > 0]$$

Thus, modulus = 2

So,

$$2 \cos \theta = -\sqrt{3} \text{ and } 2 \sin \theta = 1$$

$$\cos \theta = \frac{-\sqrt{3}}{2} \text{ and } \sin \theta = \frac{1}{2}$$

$$\therefore \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \quad [\text{As } \theta \text{ lies in the II quadrant}]$$

Therefore, the modulus and argument of the complex number $-\sqrt{3} + i$ are 2 and $\frac{5\pi}{6}$ respectively.**Convert each of the complex numbers given in Exercises 3 to 8 in the polar form:****3. $1 - i$**

Solution:

Given complex number,

$$1 - i$$

$$\text{Let } r \cos \theta = 1 \text{ and } r \sin \theta = -1$$

On squaring and adding, we get

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 1^2 + (-1)^2$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$$

$$r^2 = 2$$

$$r = \sqrt{2} = \text{Modulus [Conventionally, } r > 0]$$

So,

$$\sqrt{2} \cos \theta = 1 \text{ and } \sqrt{2} \sin \theta = -1$$

$$\cos \theta = \frac{1}{\sqrt{2}} \text{ and } \sin \theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \theta = -\frac{\pi}{4} \quad [\text{As } \theta \text{ lies in the IV quadrant}]$$

So,

$$1 - i = r \cos \theta + i r \sin \theta = \sqrt{2} \cos \left(-\frac{\pi}{4} \right) + i \sqrt{2} \sin \left(-\frac{\pi}{4} \right)$$

$$= \sqrt{2} \left[\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right]$$

Hence, this is the required polar form.

4. $-1 + i$ **Solution:**

Given complex number,

$$-1 + i$$

Let $r \cos \theta = -1$ and $r \sin \theta = 1$

On squaring and adding, we get

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = (-1)^2 + 1^2$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$$

$$r^2 = 2$$

$$r = \sqrt{2} \quad [\text{Conventionally, } r > 0]$$

So,

$$\sqrt{2} \cos \theta = -1 \text{ and } \sqrt{2} \sin \theta = 1$$

$$\cos \theta = -\frac{1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in the II quadrant}]$$

Hence, it can be written as

$$\begin{aligned} -1 + i &= r \cos \theta + i r \sin \theta = \sqrt{2} \cos \frac{3\pi}{4} + i \sqrt{2} \sin \frac{3\pi}{4} \\ &= \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \end{aligned}$$

This is the required polar form.

5. $-1 - i$

Solution:

Given complex number,

$$-1 - i$$

Let $r \cos \theta = -1$ and $r \sin \theta = -1$

On squaring and adding, we get

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = (-1)^2 + (-1)^2$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$$

$$r^2 = 2$$

$$r = \sqrt{2} \quad [\text{Conventionally, } r > 0]$$

So,

$$\sqrt{2} \cos \theta = -1 \text{ and } \sqrt{2} \sin \theta = -1$$

$$\Rightarrow \cos \theta = -\frac{1}{\sqrt{2}} \text{ and } \sin \theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \theta = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in the III quadrant}]$$

Hence, it can be written as

$$\begin{aligned} -1 - i &= r \cos \theta + i r \sin \theta = \sqrt{2} \cos \frac{-3\pi}{4} + i \sqrt{2} \sin \frac{-3\pi}{4} \\ &= \sqrt{2} \left(\cos \frac{-3\pi}{4} + i \sin \frac{-3\pi}{4} \right) \end{aligned}$$

This is the required polar form.

6. -3

Solution:

Given complex number,

-3

Let $r \cos \theta = -3$ and $r \sin \theta = 0$

On squaring and adding, we get

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = (-3)^2$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 9$$

$$r^2 = 9$$

$$r = \sqrt{9} = 3 \quad [\text{Conventionally, } r > 0]$$

So,

$$3 \cos \theta = -3 \text{ and } 3 \sin \theta = 0$$

$$\Rightarrow \cos \theta = -1 \text{ and } \sin \theta = 0$$

$$\therefore \theta = \pi$$

Hence, it can be written as

$$-3 = r \cos \theta + i r \sin \theta = 3 \cos \pi + i 3 \sin \pi = 3 (\cos \pi + i \sin \pi)$$

This is the required polar form.

7. $3 + i$

Solution:

Given complex number,

$$\sqrt{3} + i$$

Let $r \cos \theta = \sqrt{3}$ and $r \sin \theta = 1$

On squaring and adding, we get

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = (\sqrt{3})^2 + 1^2$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 3 + 1$$

$$r^2 = 4$$

$$r = \sqrt{4} = 2 \quad [\text{Conventionally, } r > 0]$$

So,

$$2 \cos \theta = \sqrt{3} \text{ and } 2 \sin \theta = 1$$

$$\Rightarrow \cos \theta = \frac{\sqrt{3}}{2} \text{ and } \sin \theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{6} \quad [\text{As } \theta \text{ lies in the I quadrant}]$$

Hence, it can be written as

$$\begin{aligned} \sqrt{3} + i &= r \cos \theta + i r \sin \theta = 2 \cos \frac{\pi}{6} + i 2 \sin \frac{\pi}{6} \\ &= 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \end{aligned}$$

This is the required polar form.

8. i

Solution:

Given complex number, i

Let $r \cos \theta = 0$ and $r \sin \theta = 1$

On squaring and adding, we get

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 0^2 + 1^2$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 1$$

$$r^2 = 1$$

$$r = \sqrt{1} = 1 \quad [\text{Conventionally, } r > 0]$$

So,

$$\cos \theta = 0 \text{ and } \sin \theta = 1$$

$$\therefore \theta = \frac{\pi}{2}$$

Hence, it can be written as

$$i = r \cos \theta + i r \sin \theta = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

This is the required polar form.

Exercise 5.3 Page No: 109

Solve each of the following equations:

1. $x^2 + 3 = 0$

Solution:

Given the quadratic equation,

$$x^2 + 3 = 0$$

On comparing it with $ax^2 + bx + c = 0$, we have

$$a = 1, b = 0, \text{ and } c = 3$$

So, the discriminant of the given equation will be

$$D = b^2 - 4ac = 0^2 - 4 \times 1 \times 3 = -12$$

Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{\pm \sqrt{-12}}{2 \times 1} = \frac{\pm \sqrt{12} i}{2} \quad [\sqrt{-1} = i]$$

$$\therefore x = \frac{\pm 2\sqrt{3} i}{2} = \pm \sqrt{3} i$$

2. $2x^2 + x + 1 = 0$

Solution:

Given the quadratic equation,

$$2x^2 + x + 1 = 0$$

On comparing it with $ax^2 + bx + c = 0$, we have

$$a = 2, b = 1, \text{ and } c = 1$$

So, the discriminant of the given equation will be

$$D = b^2 - 4ac = 1^2 - 4 \times 2 \times 1 = 1 - 8 = -7$$

Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times 2} = \frac{-1 \pm \sqrt{7}i}{4} \quad [\sqrt{-1} = i]$$

3. $x^2 + 3x + 9 = 0$

Solution:

Given the quadratic equation,

$$x^2 + 3x + 9 = 0$$

On comparing it with $ax^2 + bx + c = 0$, we have

$$a = 1, b = 3, \text{ and } c = 9$$

So, the discriminant of the given equation will be

$$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 9 = 9 - 36 = -27$$

Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-27}}{2(1)} = \frac{-3 \pm 3\sqrt{-3}}{2} = \frac{-3 \pm 3\sqrt{3}i}{2} \quad [\sqrt{-1} = i]$$

4. $-x^2 + x - 2 = 0$

Solution:

Given the quadratic equation,

$$-x^2 + x - 2 = 0$$

On comparing it with $ax^2 + bx + c = 0$, we have

$$a = -1, b = 1, \text{ and } c = -2$$

So, the discriminant of the given equation will be

$$D = b^2 - 4ac = 1^2 - 4 \times (-1) \times (-2) = 1 - 8 = -7$$

Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times (-1)} = \frac{-1 \pm \sqrt{7}i}{-2} \quad [\sqrt{-1} = i]$$

5. $x^2 + 3x + 5 = 0$

Solution:

Given the quadratic equation,

$$x^2 + 3x + 5 = 0$$

On comparing it with $ax^2 + bx + c = 0$, we have

$$a = 1, b = 3, \text{ and } c = 5$$

So, the discriminant of the given equation will be

$$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 5 = 9 - 20 = -11$$

Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-11}}{2 \times 1} = \frac{-3 \pm \sqrt{11}i}{2} \quad [\sqrt{-1} = i]$$

6. $x^2 - x + 2 = 0$

Solution:

Given the quadratic equation,

$$x^2 - x + 2 = 0$$

On comparing it with $ax^2 + bx + c = 0$, we have

$$a = 1, b = -1, \text{ and } c = 2$$

So, the discriminant of the given equation is

$$D = b^2 - 4ac = (-1)^2 - 4 \times 1 \times 2 = 1 - 8 = -7$$

Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-1) \pm \sqrt{-7}}{2 \times 1} = \frac{1 \pm \sqrt{7}i}{2} \quad [\sqrt{-1} = i]$$

7. $\sqrt{2}x^2 + x + \sqrt{2} = 0$

Solution:

Given the quadratic equation,

$$\sqrt{2}x^2 + x + \sqrt{2} = 0$$

On comparing it with $ax^2 + bx + c = 0$, we have

$$a = \sqrt{2}, b = 1, \text{ and } c = \sqrt{2}$$

So, the discriminant of the given equation is

$$D = b^2 - 4ac = (1)^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$$

Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times \sqrt{2}} = \frac{-1 \pm \sqrt{7}i}{2\sqrt{2}} \quad [\sqrt{-1} = i]$$

8. $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Solution:

Given the quadratic equation,

$$\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$$

On comparing it with $ax^2 + bx + c = 0$, we have

$$a = \sqrt{3}, b = -\sqrt{2}, \text{ and } c = 3\sqrt{3}$$

So, the discriminant of the given equation is

$$D = b^2 - 4ac = (-\sqrt{2})^2 - 4 \times \sqrt{3} \times 3\sqrt{3} = 2 - 36 = -34$$

Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-\sqrt{2}) \pm \sqrt{-34}}{2 \times \sqrt{3}} = \frac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}} \quad [\sqrt{-1} = i]$$

9. $x^2 + x + 1/\sqrt{2} = 0$

Solution:

Given the quadratic equation,

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$$x^2 + x + 1/\sqrt{2} = 0$$

It can be rewritten as,

$$\sqrt{2}x^2 + \sqrt{2}x + 1 = 0$$

On comparing it with $ax^2 + bx + c = 0$, we have

$$a = \sqrt{2}, b = \sqrt{2}, \text{ and } c = 1$$

So, the discriminant of the given equation is

$$D = b^2 - 4ac = (\sqrt{2})^2 - 4 \times \sqrt{2} \times 1 = 2 - 4\sqrt{2} = 2(1 - 2\sqrt{2})$$

Hence, the required solutions are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{D}}{2a} = \frac{-\sqrt{2} \pm \sqrt{2-4\sqrt{2}}}{2 \times \sqrt{2}} = \frac{-\sqrt{2} \pm \sqrt{2(1-2\sqrt{2})}}{2\sqrt{2}} \\ &= \left(\frac{-\sqrt{2} \pm \sqrt{2}(\sqrt{2\sqrt{2}-1})i}{2\sqrt{2}} \right) \quad [\sqrt{-1} = i] \\ &= \frac{-1 \pm (\sqrt{2\sqrt{2}-1})i}{2} \end{aligned}$$

10. $x^2 + x/\sqrt{2} + 1 = 0$

Solution:

Given the quadratic equation,

$$x^2 + x/\sqrt{2} + 1 = 0$$

It can be rewritten as,

$$\sqrt{2}x^2 + x + \sqrt{2} = 0$$

On comparing it with $ax^2 + bx + c = 0$, we have

$$a = \sqrt{2}, b = 1, \text{ and } c = \sqrt{2}$$

So, the discriminant of the given equation is

$$D = b^2 - 4ac = (1)^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$$

Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2\sqrt{2}} = \frac{-1 \pm \sqrt{7}i}{2\sqrt{2}} \quad [\sqrt{-1} = i]$$

Miscellaneous Exercise Page No: 112

1.

Evaluate: $\left[i^{18} + \left(\frac{1}{i} \right)^{25} \right]^3$

Solution:

$$\begin{aligned}
 & \left[i^{18} + \left(\frac{1}{i} \right)^{25} \right]^3 \\
 &= \left[i^{4 \times 4 + 2} + \frac{1}{i^{4 \times 6 + 1}} \right]^3 \\
 &= \left[(i^4)^4 \cdot i^2 + \frac{1}{(i^4)^6 \cdot i} \right]^3 \\
 &= \left[i^2 + \frac{1}{i} \right]^3 \quad [i^4 = 1] \\
 &= \left[-1 + \frac{1}{i} \times \frac{i}{i} \right]^3 \quad [i^2 = -1] \\
 &= \left[-1 + \frac{i}{i^2} \right]^3 \\
 &= [-1 - i]^3 \\
 &= (-1)^3 [1 + i]^3 \\
 &= -[1^3 + i^3 + 3 \cdot 1 \cdot i(1 + i)] \\
 &= -[1 + i^3 + 3i + 3i^2] \\
 &= -[1 - i + 3i - 3] \\
 &= -[-2 + 2i] \\
 &= 2 - 2i
 \end{aligned}$$

2. For any two complex numbers z_1 and z_2 , prove that

$$\operatorname{Re}(z_1 z_2) = \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2$$

Solution:

Lets's assume $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ as two complex numbers

Product of these complex numbers, $z_1 z_2$

$$\begin{aligned}
 z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\
 &= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2) \\
 &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 \\
 &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 - y_1 y_2 \quad [i^2 = -1] \\
 &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)
 \end{aligned}$$

Now,

$$\operatorname{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2$$

$$\Rightarrow \operatorname{Re}(z_1 z_2) = \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2$$

Hence, proved.

3. Reduce to the standard form.

$$\left(\frac{1}{1-4i} - \frac{2}{1+i} \right) \left(\frac{3-4i}{5+i} \right)$$

Solution:

$$\begin{aligned} \left(\frac{1}{1-4i} - \frac{2}{1+i} \right) \left(\frac{3-4i}{5+i} \right) &= \left[\frac{(1+i) - 2(1-4i)}{(1-4i)(1+i)} \right] \left[\frac{3-4i}{5+i} \right] \\ &= \left[\frac{1+i-2+8i}{1+i-4i-4i^2} \right] \left[\frac{3-4i}{5+i} \right] = \left[\frac{-1+9i}{5-3i} \right] \left[\frac{3-4i}{5+i} \right] \\ &= \left[\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2} \right] = \frac{33+31i}{28-10i} = \frac{33+31i}{2(14-5i)} \end{aligned}$$

$$= \frac{(33+31i)}{2(14-5i)} \times \frac{(14+5i)}{(14+5i)} \quad [\text{On multiplying numerator and denominator by } (14+5i)]$$

$$= \frac{462+165i+434i+155i^2}{2[(14)^2-(5i)^2]} = \frac{307+599i}{2(196-25i^2)}$$

$$= \frac{307+599i}{2(221)} = \frac{307+599i}{442} = \frac{307}{442} + \frac{599i}{442}$$

Hence, this is the required standard form.

4.

If $x - iy = \sqrt{\frac{a-ib}{c-id}}$ prove that $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$.

Solution:

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Given,

$$\begin{aligned} x - iy &= \sqrt{\frac{a - ib}{c - id}} \\ &= \sqrt{\frac{a - ib}{c - id} \times \frac{c + id}{c + id}} \quad [\text{On multiplying numerator and denominator by } (c + id)] \\ &= \sqrt{\frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}} \end{aligned}$$

So,

$$(x - iy)^2 = \frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}$$

$$x^2 - y^2 - 2ixy = \frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}$$

On comparing real and imaginary parts, we get

$$x^2 - y^2 = \frac{ac + bd}{c^2 + d^2}, \quad -2xy = \frac{ad - bc}{c^2 + d^2} \quad (1)$$

$$\begin{aligned} (x^2 + y^2)^2 &= (x^2 - y^2)^2 + 4x^2y^2 \\ &= \left(\frac{ac + bd}{c^2 + d^2}\right)^2 + \left(\frac{ad - bc}{c^2 + d^2}\right)^2 \quad [\text{Using (1)}] \\ &= \frac{a^2c^2 + b^2d^2 + 2acbd + a^2d^2 + b^2c^2 - 2adbc}{(c^2 + d^2)^2} \\ &= \frac{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}{(c^2 + d^2)^2} \\ &= \frac{a^2(c^2 + d^2) + b^2(c^2 + d^2)}{(c^2 + d^2)^2} \\ &= \frac{(c^2 + d^2)(a^2 + b^2)}{(c^2 + d^2)^2} \\ &= \frac{a^2 + b^2}{c^2 + d^2} \end{aligned}$$

- Hence Proved

5. Convert the following into the polar form:

(i) $\frac{1+7i}{(2-i)^2}$, (ii) $\frac{1+3i}{1-2i}$

Solution:

$$\begin{aligned}
 \text{(i) Here, } z &= \frac{1+7i}{(2-i)^2} \\
 &= \frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i} = \frac{1+7i}{4-1-4i} \\
 &= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i+21i+28i^2}{3^2+4^2} \quad [\text{Multiplying by its conjugate in the numerator and denominator}] \\
 &= \frac{3+4i+21i-28}{3^2+4^2} = \frac{-25+25i}{25} \\
 &= -1+i
 \end{aligned}$$

Let $r \cos \theta = -1$ and $r \sin \theta = 1$

On squaring and adding, we get

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 2$$

$$r^2 = 2 \quad [\cos^2 \theta + \sin^2 \theta = 1]$$

$$r = \sqrt{2} \quad [\text{Conventionally, } r > 0]$$

So,

$$\sqrt{2} \cos \theta = -1 \text{ and } \sqrt{2} \sin \theta = 1$$

$$\Rightarrow \cos \theta = \frac{-1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in II quadrant}]$$

Expressing as, $z = r \cos \theta + i r \sin \theta$

$$= \sqrt{2} \cos \frac{3\pi}{4} + i \sqrt{2} \sin \frac{3\pi}{4} = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

Therefore, this is the required polar form.

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$$\begin{aligned}
 \text{(ii) Let, } z &= \frac{1+3i}{1-2i} \\
 &= \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i} \\
 &= \frac{1+2i+3i-6}{1+4} \\
 &= \frac{-5+5i}{5} = -1+i
 \end{aligned}$$

Now,

$$\text{Let } r \cos \theta = -1 \text{ and } r \sin \theta = 1$$

On squaring and adding, we get

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 2$$

$$r^2 = 2 \quad [\cos^2 \theta + \sin^2 \theta = 1]$$

$$\Rightarrow r = \sqrt{2} \quad [\text{Conventionally, } r > 0]$$

$$\therefore \sqrt{2} \cos \theta = -1 \text{ and } \sqrt{2} \sin \theta = 1$$

$$\cos \theta = \frac{-1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in II quadrant}]$$

Expressing as, $z = r \cos \theta + i r \sin \theta$

$$z = \sqrt{2} \cos \frac{3\pi}{4} + i \sqrt{2} \sin \frac{3\pi}{4} = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

Therefore, this is the required polar form.

Solve each of the equations in Exercises 6 to 9.

6. $3x^2 - 4x + 20/3 = 0$

Solution:

Given the quadratic equation, $3x^2 - 4x + 20/3 = 0$

It can be re-written as: $9x^2 - 12x + 20 = 0$

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = 9, b = -12, \text{ and } c = 20$$

So, the discriminant of the given equation will be

$$D = b^2 - 4ac = (-12)^2 - 4 \times 9 \times 20 = 144 - 720 = -576$$

Hence, the required solutions are

$$\begin{aligned}
 x &= \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-12) \pm \sqrt{-576}}{2 \times 9} = \frac{12 \pm \sqrt{576} i}{18} \\
 &= \frac{12 \pm 24i}{18} = \frac{6(2 \pm 4i)}{18} = \frac{2 \pm 4i}{3} = \frac{2}{3} \pm \frac{4}{3}i
 \end{aligned}$$

7. $x^2 - 2x + 3/2 = 0$

Solution:

Given the quadratic equation, $x^2 - 2x + 3/2 = 0$

It can be re-written as $2x^2 - 4x + 3 = 0$

On comparing it with $ax^2 + bx + c = 0$, we get

$a = 2$, $b = -4$, and $c = 3$

So, the discriminant of the given equation will be

$$D = b^2 - 4ac = (-4)^2 - 4 \times 2 \times 3 = 16 - 24 = -8$$

Hence, the required solutions are

$$\begin{aligned}x &= \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-4) \pm \sqrt{-8}}{2 \times 2} = \frac{4 \pm 2\sqrt{2}i}{4} \quad [\sqrt{-1} = i] \\&= \frac{2 \pm \sqrt{2}i}{2} = 1 \pm \frac{\sqrt{2}}{2}i\end{aligned}$$

8. $27x^2 - 10x + 1 = 0$

Solution:

Given the quadratic equation, $27x^2 - 10x + 1 = 0$

On comparing it with $ax^2 + bx + c = 0$, we get

$a = 27$, $b = -10$, and $c = 1$

So, the discriminant of the given equation will be

$$D = b^2 - 4ac = (-10)^2 - 4 \times 27 \times 1 = 100 - 108 = -8$$

Hence, the required solutions are

$$\begin{aligned}x &= \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-10) \pm \sqrt{-8}}{2 \times 27} = \frac{10 \pm 2\sqrt{2}i}{54} \\&= \frac{5 \pm \sqrt{2}i}{27} = \frac{5}{27} \pm \frac{\sqrt{2}}{27}i\end{aligned}$$

9. $21x^2 - 28x + 10 = 0$

Solution:

Given the quadratic equation, $21x^2 - 28x + 10 = 0$

On comparing it with $ax^2 + bx + c = 0$, we have

$a = 21$, $b = -28$, and $c = 10$

So, the discriminant of the given equation will be

$$D = b^2 - 4ac = (-28)^2 - 4 \times 21 \times 10 = 784 - 840 = -56$$

Hence, the required solutions are

$$\begin{aligned}x &= \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-28) \pm \sqrt{-56}}{2 \times 21} = \frac{28 \pm \sqrt{56}i}{42} \\&= \frac{28 \pm 2\sqrt{14}i}{42} = \frac{28}{42} \pm \frac{2\sqrt{14}}{42}i = \frac{2}{3} \pm \frac{\sqrt{14}}{21}i\end{aligned}$$

10. If $z_1 = 2 - i$, $z_2 = 1 + i$, find

$$\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right|$$

Solution:

Given, $z_1 = 2 - i$, $z_2 = 1 + i$

So,

$$\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right| = \left| \frac{(2-i) + (1+i) + 1}{(2-i) - (1+i) + 1} \right|$$

$$= \left| \frac{4}{2-2i} \right| = \left| \frac{4}{2(1-i)} \right|$$

$$= \left| \frac{2}{1-i} \times \frac{1+i}{1+i} \right| = \left| \frac{2(1+i)}{1^2 - i^2} \right|$$

$$= \left| \frac{2(1+i)}{1+1} \right| \quad [i^2 = -1]$$

$$= \left| \frac{2(1+i)}{2} \right|$$

$$= |1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Hence, the value of $\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right|$ is $\sqrt{2}$.

11.

If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$.

Solution:

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Given,

$$\begin{aligned}a + ib &= \frac{(x+i)^2}{2x^2+1} \\&= \frac{x^2+i^2+2xi}{2x^2+1} \\&= \frac{x^2-1+i2x}{2x^2+1} \\&= \frac{x^2-1}{2x^2+1} + i\left(\frac{2x}{2x^2+1}\right)\end{aligned}$$

Comparing the real and imaginary parts, we have

$$\begin{aligned}a &= \frac{x^2-1}{2x^2+1} \text{ and } b = \frac{2x}{2x^2+1} \\ \therefore a^2 + b^2 &= \left(\frac{x^2-1}{2x^2+1}\right)^2 + \left(\frac{2x}{2x^2+1}\right)^2 \\&= \frac{x^4+1-2x^2+4x^2}{(2x^2+1)^2} \\&= \frac{x^4+1+2x^2}{(2x^2+1)^2} \\&= \frac{(x^2+1)^2}{(2x^2+1)^2}\end{aligned}$$

Hence proved,

$$a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$

12. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

(i) $\operatorname{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$, (ii) $\operatorname{Im}\left(\frac{1}{z_1 \bar{z}_1}\right)$

Solution:

Given,

$$z_1 = 2 - i, z_2 = -2 + i$$

$$(i) z_1 z_2 = (2 - i)(-2 + i) = -4 + 2i + 2i - i^2 = -4 + 4i - (-1) = -3 + 4i$$

$$\bar{z}_1 = 2 + i$$

$$\therefore \frac{z_1 z_2}{\bar{z}_1} = \frac{-3 + 4i}{2 + i}$$

On multiplying numerator and denominator by $(2 - i)$, we get

$$\begin{aligned} \frac{z_1 z_2}{\bar{z}_1} &= \frac{(-3 + 4i)(2 - i)}{(2 + i)(2 - i)} = \frac{-6 + 3i + 8i - 4i^2}{2^2 + 1^2} = \frac{-6 + 11i - 4(-1)}{2^2 + 1^2} \\ &= \frac{-2 + 11i}{5} = \frac{-2}{5} + \frac{11}{5}i \end{aligned}$$

Comparing the real parts, we have

$$\operatorname{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = \frac{-2}{5}$$

$$(ii) \frac{1}{z_1 \bar{z}_1} = \frac{1}{(2 - i)(2 + i)} = \frac{1}{(2)^2 + (1)^2} = \frac{1}{5}$$

On comparing the imaginary part, we get

$$\operatorname{Im}\left(\frac{1}{z_1 \bar{z}_1}\right) = 0$$

13. Find the modulus and argument of the complex number.

$$\frac{1 + 2i}{1 - 3i}$$

Solution:

Let $z = \frac{1+2i}{1-3i}$, then

$$z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1^2+3^2} = \frac{1+5i+6(-1)}{1+9}$$

$$= \frac{-5+5i}{10} = \frac{-5}{10} + \frac{5i}{10} = \frac{-1}{2} + \frac{1}{2}i$$

Let $z = r \cos \theta + ir \sin \theta$

So,

$$r \cos \theta = \frac{-1}{2} \text{ and } r \sin \theta = \frac{1}{2}$$

On squaring and adding, we get

$$r^2 (\cos^2 \theta + \sin^2 \theta) = \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$r^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad [\text{Conventionally, } r > 0]$$

$$r = \frac{1}{\sqrt{2}}$$

Now,

$$\frac{1}{\sqrt{2}} \cos \theta = \frac{-1}{2} \text{ and } \frac{1}{\sqrt{2}} \sin \theta = \frac{1}{2}$$

$$\Rightarrow \cos \theta = \frac{-1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in the II quadrant}]$$

14. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

Solution:

Let's assume $z = (x - iy)(3 + 5i)$

$$z = 3x + 5xi - 3yi - 5yi^2 = 3x + 5xi - 3yi + 5y = (3x + 5y) + i(5x - 3y)$$

$$\therefore \bar{z} = (3x + 5y) - i(5x - 3y)$$

Also given, $\bar{z} = -6 - 24i$

And,

$$(3x + 5y) - i(5x - 3y) = -6 - 24i$$

On equating real and imaginary parts, we have

$$3x + 5y = -6 \dots\dots (i)$$

$$5x - 3y = 24 \dots\dots (ii)$$

Performing (i) $\times 3$ + (ii) $\times 5$, we get

$$(9x + 15y) + (25x - 15y) = -18 + 120$$

$$34x = 102$$

$$x = 102/34 = 3$$

Putting the value of x in equation (i), we get

$$3(3) + 5y = -6$$

$$5y = -6 - 9 = -15$$

$$y = -3$$

Therefore, the values of x and y are 3 and -3, respectively.

15. Find the modulus of

$$\frac{1+i}{1-i} - \frac{1-i}{1+i}$$

Solution:

$$\begin{aligned}\frac{1+i}{1-i} - \frac{1-i}{1+i} &= \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} \\ &= \frac{1+i^2+2i-1-i^2+2i}{1^2+1^2} \\ &= \frac{4i}{2} = 2i\end{aligned}$$

$$\therefore \left| \frac{1+i}{1-i} - \frac{1-i}{1+i} \right| = |2i| = \sqrt{2^2} = 2$$

16. If $(x + iy)^3 = u + iv$, then show that

$$\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$$

Solution:

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Given,

$$(x + iy)^3 = u + iv$$

$$x^3 + (iy)^3 + 3 \cdot x \cdot iy(x + iy) = u + iv$$

$$x^3 + i^3 y^3 + 3x^2 yi + 3xy^2 i^2 = u + iv$$

$$x^3 - iy^3 + 3x^2 yi - 3xy^2 = u + iv$$

$$(x^3 - 3xy^2) + i(3x^2 y - y^3) = u + iv$$

On equating real and imaginary parts, we get

$$u = x^3 - 3xy^2, \quad v = 3x^2 y - y^3$$

$$\begin{aligned} \frac{u}{x} + \frac{v}{y} &= \frac{x^3 - 3xy^2}{x} + \frac{3x^2 y - y^3}{y} \\ &= \frac{x(x^2 - 3y^2)}{x} + \frac{y(3x^2 - y^2)}{y} \\ &= x^2 - 3y^2 + 3x^2 - y^2 \\ &= 4x^2 - 4y^2 \\ &= 4(x^2 - y^2) \end{aligned}$$

$$\therefore \frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$$

Hence proved.

17. If α and β are different complex numbers with $|\beta| = 1$, then find

$$\frac{|\beta - \alpha|}{|1 - \bar{\alpha}\beta|}$$

Solution:

Let $\alpha = a + ib$ and $\beta = x + iy$

Given, $|\beta| = 1$

$$\text{So, } \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow x^2 + y^2 = 1 \quad \dots (i)$$

$$\begin{aligned} \left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| &= \left| \frac{(x + iy) - (a + ib)}{1 - (a - ib)(x + iy)} \right| \\ &= \left| \frac{(x - a) + i(y - b)}{1 - (ax + aiy - ibx + by)} \right| \\ &= \left| \frac{(x - a) + i(y - b)}{(1 - ax - by) + i(bx - ay)} \right| \\ &= \frac{|(x - a) + i(y - b)|}{|(1 - ax - by) + i(bx - ay)|} \quad \left[\frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|} \right] \\ &= \frac{\sqrt{(x - a)^2 + (y - b)^2}}{\sqrt{(1 - ax - by)^2 + (bx - ay)^2}} \\ &= \frac{\sqrt{x^2 + a^2 - 2ax + y^2 + b^2 - 2by}}{\sqrt{1 + a^2x^2 + b^2y^2 - 2ax + 2abxy - 2by + b^2x^2 + a^2y^2 - 2abxy}} \\ &= \frac{\sqrt{(x^2 + y^2) + a^2 + b^2 - 2ax - 2by}}{\sqrt{1 + a^2(x^2 + y^2) + b^2(y^2 + x^2) - 2ax - 2by}} \\ &= \frac{\sqrt{(x^2 + y^2) + a^2 + b^2 - 2ax - 2by}}{\sqrt{1 + a^2(x^2 + y^2) + b^2(y^2 + x^2) - 2ax - 2by}} \\ &= \frac{\sqrt{1 + a^2 + b^2 - 2ax - 2by}}{\sqrt{1 + a^2 + b^2 - 2ax - 2by}} \\ &= 1 \end{aligned}$$

$$\therefore \left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = 1$$

18. Find the number of non-zero integral solutions of the equation $|1 - i|^x = 2^x$

Solution:

$$|1 - i|^x = 2^x$$

$$\left(\sqrt{1^2 + (-1)^2}\right)^x = 2^x$$

$$(\sqrt{2})^x = 2^x$$

$$2^{\frac{x}{2}} = 2^x$$

$$\frac{x}{2} = x$$

$$x = 2x$$

$$2x - x = 0$$

$$x = 0$$

Therefore, 0 is the only integral solution of the given equation.

Hence, the number of non-zero integral solutions of the given equation is 0.

19. If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that
 $(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$

Solution:

Given,

$$(a + ib)(c + id)(e + if)(g + ih) = A + iB$$

$$\therefore |(a + ib)(c + id)(e + if)(g + ih)| = |A + iB|$$

$$\Rightarrow |(a + ib)| \times |(c + id)| \times |(e + if)| \times |(g + ih)| = |A + iB| \quad [|z_1 z_2| = |z_1| |z_2|]$$

$$\sqrt{a^2 + b^2} \times \sqrt{c^2 + d^2} \times \sqrt{e^2 + f^2} \times \sqrt{g^2 + h^2} = \sqrt{A^2 + B^2}$$

On squaring both sides, we get

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$$

Hence proved.

20. If, then find the least positive integral value of m .

$$\left(\frac{1+i}{1-i}\right)^m = 1$$

Solution:

$$\left(\frac{1+i}{1-i}\right)^m = 1$$

$$\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = 1$$

$$\left(\frac{(1+i)^2}{1^2+1^2}\right)^m = 1$$

$$\left(\frac{1^2+i^2+2i}{2}\right)^m = 1$$

$$\left(\frac{1-1+2i}{2}\right)^m = 1$$

$$\left(\frac{2i}{2}\right)^m = 1$$

$$i^m = 1$$

Hence, $m = 4k$, where k is some integer.

Thus, the least positive integer is 1.

Therefore, the least positive integral value of m is 4 ($= 4 \times 1$)

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